Noncommutative fermions and Morita equivalence

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Abstract

We study the Morita equivalence for fermion theories on noncommutative two-tori. For rational values of the θ parameter (in appropriate units) we show the equivalence between an abelian noncommutative fermion theory and a nonabelian theory of twisted fermions on ordinary space. We study the chiral anomaly and compute the determinant of the Dirac operator in the dual theories showing that the Morita equivalence also holds at this level.

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1 Introduction

Noncommutative Field Theories (NCFT) have attracted much attention in the last years because they naturally arise as some low energy limit of open string theory and as the compactification of M-theory on the torus [1],[2].

A significant feature of the noncommutative field theory is the Morita duality between noncommutative tori. This duality is a powerful mathematical result that establishes a relation, via an isomorphism, between two noncommutative algebras. Of particular importance are the algebras defined on the noncommutative torus, where it can be shown that are Morita equivalent if the corresponding sizes of the tori and the noncommutative parameters are related in a specific way.

There has been several results in the literature about the Morita equivalence of NCFT but principally focused on noncommutative gauge theories and describing mostly classical or semiclassical aspects of them [3]-[12]. In particular there has been relatively very little work on other than gauge theories or in the quantum aspects of the equivalence. Central questions as if there are “Morita anomalies” are still open.

In this work we want to fill some gaps on the subject. First we are going to establish the Morita equivalence for fermion theories. We are going to show that there is a well defined isomorphism between the correlation functions of fermions on a noncommutative torus and those of a non-abelian fermion theory on ordinary space. Also we are going to analize the effect of the Morita map on the chiral anomaly and compute and compare the fermionic determinant of dual theories. Finally we are going to discuss the bosonization of fermion theories defined on dual noncommutative tori.

2 Morita equivalence for fermionic fields

In this section we will explicitly construct the Morita equivalence for fermionic theories for some rational values of the noncommutative parameter. The Morita equivalence is an isomorphism between noncommutative algebras that conserves all the modules and their associated structures. Let us consider the noncommutative torus T^2_θ and for simplicity, of radii R. The coordinates satisfy the commutation rule

$$[x_1, x_2] = i\theta \tag{1}$$

An associative algebra of smooth functions over T^2_θ can be realized through the Moyal product

$$f(x) * g(x) = \exp\left(\frac{i\theta}{2}(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})\right)\ f(x)\ g(y)\Big|_{y=x} \tag{2}$$

It is convenient to decompose the elements of the algebra in their Fourier components. However, when dealing with fermions defined on a torus we must be aware that they can have different spin structures associated to any of the compact directions. For the torus we can have four different spin structures characterized as follows

$$\begin{aligned}\psi_(x_1 + R, x_2) &= e^{2\pi i\alpha_1}\ \psi(x_1, x_2)\\ \psi_(x_1, x_2 + R) &= e^{2\pi i\alpha_2}\ \psi(x_1, x_2)\end{aligned} \tag{3}$$

where α_1 and α_2 can take the values $0, 1/2$. We will call a fermion with boundary conditions (3) as of type $\vec{\alpha} = (\alpha_1, \alpha_2)$ and denote it $\psi_{\vec{\alpha}}$. Fermions with $\alpha_i = 0$ $(i = 1, 2)$ are called Ramond (R) and with $\alpha_i = 1/2$ are called Neveu-Schwarz (NS). So the possible spin structures for the two-torus are R-R, R-NS, NS-R and NS-NS.

The Fourier expansion of a fermion field of type $\vec{\alpha}$ has the following form:

$$\psi_{\vec{\alpha}} = \sum_{\vec{k}} \psi^{\vec{k}} U_{\vec{k}+\vec{\alpha}} \tag{4}$$

where we have defined the generators

$$U_{\vec{k}+\vec{\alpha}} \equiv \exp\Big(2\pi i(\vec{k} + \vec{\alpha}) \cdot \vec{x}/R\Big) \tag{5}$$

The Moyal commutator of the generators can be easily computed to give

$$[U_{\vec{k}+\vec{\alpha}}, U_{\vec{k}'+\vec{\alpha}'}] = -2i \sin\left(\frac{2\pi^2\theta}{R^2}(\vec{k} + \vec{\alpha}) \wedge (\vec{k}' + \vec{\alpha}')\right) U_{\vec{k}+\vec{k}'+\vec{\alpha}+\vec{\alpha}'} \tag{6}$$

where $\vec{p} \wedge \vec{q} = \varepsilon_{ij}\ p_i q_j$.

When the noncommutative parameter θ takes a the value

$$\theta = \frac{4M}{N}\frac{R^2}{2\pi} \tag{7}$$

being M and N relatively prime integers, an interesting feature of the algebra generated by the $U_{\vec{k}+\vec{\alpha}}$ emerges. First, the infinite-dimensional algebra breaks up into equivalence classes of finite dimensional subspaces. Indeed, noticing that the elements

$$U_{N\vec{k}/2} \tag{8}$$

generates the center of the algebra, we can decompose the momenta in the form

$$2(\vec{k}' + \vec{\alpha}) = N\vec{k} + \vec{n} \;, \qquad 0 \le n_1 \le N \;, \;\; 0 \le n_2 \le N \tag{9}$$

and the whole algebra splits into equivalence classes classified by the all possible values of $N\vec{k}$. Each class is itself a subalgebra generated by the N^2 functions

$$U_{\vec{n}+\vec{\alpha}} \;, \qquad 0 \le n_1 \le N \;, \;\; 0 \le n_2 \le N \tag{10}$$

satisfying

$$[U_{\vec{n}+\vec{\alpha}}, U_{\vec{n}'+\vec{\alpha}'}] = -2i \sin\left(\pi \frac{M}{N}(2\vec{k} + 2\vec{\alpha}) \wedge (2\vec{k}' + 2\vec{\alpha}')\right) U_{\vec{n}+\vec{n}'+\vec{\alpha}+\vec{\alpha}'} \tag{11}$$

It is easy to show that the algebra (11) is isomorphic to the (complexification of the) Lie algebra $u(N)$. A N-dimensional representation of this algebra can be constructed with 't Hofft's "shift" and "clock" matrices [9],[10],[13]

$$Q = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \omega^{N-1} \end{pmatrix} \;, \qquad P = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & & \cdot & \cdot \\ & & & & \cdot & 1 \\ 1 & & & & 0 \end{pmatrix} \tag{12}$$

where $\omega = \exp\left(\frac{2\pi i M}{N}\right)$. Indeed, the matrices

$$J_{\vec{n}} = \omega^{\frac{n_1 n_2}{2}} Q^{n_1} P^{n_2} \;, \qquad n_1 = 0, \cdots, N-1 \;, \;\; n_2 = 0, \cdots, N-1 \tag{13}$$

generates an algebra isomorphic to (11)

$$[J_{\vec{n}}, J_{\vec{m}}] = -2i \sin\left(\pi \frac{M}{N} \vec{n} \wedge \vec{m}\right) J_{\vec{n}+\vec{m}} \tag{14}$$

Thus, we have a map (Morita mapping) between the Fourier modes defined on a noncommutative torus and functions taking values on $u(N)$ defined on a commutative space:

$$\exp\left(2\pi i(\vec{k} + \vec{\alpha}) \cdot \hat{\vec{x}}/R)\right) \leftrightarrow \exp\left(2\pi i(\vec{k} + \vec{\alpha}) \cdot \vec{x}/R\right) J_{2\vec{k}+2\vec{\alpha}} \tag{15}$$

This mapping generates a mapping between fermion fields in the following way. For the sake of simplicity, let us consider the case $N = 2N'$ in which the decomposing of the momenta is

$$\vec{k} = N'\vec{q} + \vec{n} \ , \qquad 0 \leq n_1 \leq N' \ , \ \ 0 \leq n_2 \leq N' \tag{16}$$

Then, we write the fermion field on the noncommutative torus T^2_θ with spin structure $\vec{\alpha}$ in the form

$$\psi_{\vec{\alpha}} = \sum_{\vec{q}} \exp\left(2\pi i N'\vec{q}\cdot\vec{x}/R\right) \sum_{\vec{n}=0}^{N'-1} \psi^{\vec{q},\vec{n}} U_{\vec{n}+\alpha} \tag{17}$$

Now, using (15) is immediate to see that the Morita correspondence between fermion fields is given by

$$\psi_\alpha \quad \leftrightarrow \quad \psi \tag{18}$$

where

$$\psi = \sum_{\vec{n}=0}^{N'-1} \chi^{(\vec{n})} J_{2\vec{n}+2\vec{\alpha}} \tag{19}$$

and we have defined

$$\chi^{(\vec{n})} = \exp\left(2\pi i(\vec{n}+\vec{\alpha})\cdot\vec{x}/R\right) \sum_{\vec{q}} \psi^{\vec{q},\vec{n}} \exp(2\pi i N'\vec{q}\cdot\vec{x}/R) \tag{20}$$

Notice that the fermion ψ is defined in the dual torus of size $R' = R/N'$ satisfying the boundary conditions

$$\begin{aligned} \psi(x_1 + R', x_2) &= \Omega_1^\dagger \cdot \psi(x_1, x_2) \cdot \Omega_1 \\ \psi(x_1, x_2 + R') &= \Omega_2^\dagger \cdot \psi(x_1, x_2) \cdot \Omega_2 \end{aligned} \tag{21}$$

with

$$\Omega_1 = P^b, \quad \Omega_2 = Q^{1/M} \tag{22}$$

where b is an integer satisfying $aN - bM = 1$. While the field components $\chi^{(\vec{n})}$ obey twisted boundary conditions

$$\begin{aligned} \chi^{(\vec{n})}(x_1 + R', x_2) &= e^{2\pi i\,(n_1+\alpha_1)/N'} \chi^{(\vec{n})}(x_1, x_2) \\ \chi^{(\vec{n})}(x_1, x_2 + R') &= e^{2\pi i\,(n_2+\alpha_2)/N'} \chi^{(\vec{n})}(x_1, x_2) \end{aligned} \tag{23}$$

That is, the spin fields $\chi^{(\vec{n})}$ have (twisted) spin structures

$$\left(\frac{n_1+\alpha_1}{N'},\frac{n_2+\alpha_2}{N'}\right) \tag{24}$$

Let us remark that integrating over the noncommutative torus T^2_θ is equivalent to taking trace in the group U(N) and integrating over the dual torus $T^{2'}$ simultaneously

$$\int_{T^2_\theta} \hat{\Gamma}(\vec{x})\, d^2x = \frac{N'}{2}\mathrm{tr}_G \int_{T^{2'}} \Gamma(\vec{x})\, d^2x \tag{25}$$

3 Free fermions fields on the torus

Consider, as warm up, a theory of a free Dirac fermion on the noncommutative torus T^2_θ. The action is given by

$$S = -\frac{1}{8\pi}\int_{T^2_\theta} d^2x\, \bar{\psi} * (\not{\partial} + m)\psi \tag{26}$$

Notice that being the action quadratic in the fields, the Moyal product can be replaced with the ordinary product and the action is insensitive to θ.

For the case $2\pi\theta/R^2 = 4M/2N'$, we can use the Morita map (15),(25) and we get

$$S = -\frac{N'^2}{8\pi}\sum_{\vec{n}=0}^{N'-1}\int_{T^{2'}} d^2x\, \bar{\chi}^{(\vec{n})}(\not{\partial} + m)\chi^{(\vec{n})} \tag{27}$$

where T'^2 is the commutative torus of radii $(R/N', R/N')$. Thus, the Morita equivalence establishes the relation between a theory of fermions with spin structure $\vec{\alpha}$ defined on the noncommutative torus T^2_θ and a theory of N'^2 fermions with spins structures

$$\left(\frac{n_1+\alpha_1}{N'},\frac{n_2+\alpha_2}{N'}\right)\ , \quad n_1 = 1,\cdots,N'-1\ ,\ \ n_2 = 1,\cdots,N'-1 \tag{28}$$

defined on a commutative torus T'^2. Notice that the action (26) is quadratic in the fields, and consequently independent of θ. Thus the equivalence between actions (26) and (27) is *independent* of N'.

At this point we have worked at classical level. We will see that the equivalence also works at quantum level.

First we compute the partition function of both theories. As we stated above, the action (26) is independent of θ, so the $*$ product can be replaced by the ordinary product. Hence, the partition function is the one of a free Dirac fermion on an ordinary torus T^2, *i.e.*

$$Z = \det{}_{T^2}\left(\not{\partial} + m\right) \tag{29}$$

This determinant can be computed exactly for the case $m = 0$, where the theory becomes conformally invariant, and is given by [14], [15] [1]

$$Z = \frac{\left|\vartheta\begin{bmatrix}\alpha_1 - 1/2\\ \alpha_2 - 1/2\end{bmatrix}(0,\tau)\right|^2}{|\eta(\tau)|^2} \tag{30}$$

where

$$\begin{aligned}\vartheta\begin{bmatrix}a\\ b\end{bmatrix}(z,\tau) &= \sum_{n\in\mathbb{Z}} \exp\{i\pi\tau(n+a)^2 + 2\pi i(n+a)(z+b)^2\}\ ,\\ \eta(\tau) &= e^{i\frac{\pi\tau}{12}}\prod_{n=1}^{\infty}\left(1 - e^{2\pi i n\tau}\right)\end{aligned}$$

are the Jacobi theta function and the Dedekind eta function respectively, and $\tau = i$, the imaginary unity.

Now we compute the partition function of the theory defined by (27). This theory corresponds to N'^2 free fermions with the spin structures given in equation (28). The partition function is given by

$$Z' = \prod_{n_1,n_2=0}^{N'-1}\left|\frac{\vartheta\begin{bmatrix}(\alpha_1+n_1)/N - 1/2\\ (\alpha_2+n_2)/N - 1/2\end{bmatrix}(0,\tau)}{\eta(\tau)}\right|^2 \tag{31}$$

Remarkably the partitions functions (30) and (31) turn out to be identical (we checked this fact numerically but it seems that the analytical proof is a generalization of the Riemann's theta relations).

Again, we would like to stress that the identity $Z = Z'$ is independent of N.

[1] Strictly speaking this result is valid only for $\vec{\alpha} \neq (0,0)$. If $\alpha = (0,0)$ the Dirac operator has a zero mode that has to be eliminated. For simplicity from now on we will only consider the case $\vec{\alpha} \neq (0,0)$.

Now, let us concentrate on Green functions of both theories. It is interesting to consider Green functions of local operators in the noncommutative torus and θ dependent. Thus, consider the following mass operators

$$\hat{S}_1(x) = \bar{\psi}_R(x) * \psi_L(x) \ , \quad \hat{S}_2(x) = \bar{\psi}_L(x) * \psi_R(x) \tag{32}$$

and the v.e.v.

$$\langle \hat{S}_1(x)\ \hat{S}_2(y) \rangle = \langle \bar{\psi}_R(x) * \psi_L(x)\ \bar{\psi}_L(y) * \psi_R(y) \rangle \tag{33}$$

This v.e.v. can be written in the form

$$\begin{aligned} \langle \hat{S}_1(\vec{x})\ \hat{S}_2(\vec{y}) \rangle \ &= \ \exp\left(\frac{i\theta}{2} \left(\partial_{x_1}\partial_{x'_2} - \partial_{x_2}\partial_{x'_1} + \partial_{y_1}\partial_{y'_2} - \partial_{y_2}\partial_{y'_1} \right) \right) \\ &\qquad \langle \psi_L(\vec{x}')\bar{\psi}_L(\vec{y}) \rangle \big|_{\vec{x}'=\vec{x}} \quad \langle \psi_R(\vec{y}')\bar{\psi}_R(\vec{x}) \rangle \big|_{\vec{y}'=\vec{y}} \end{aligned} \tag{34}$$

The two-point function is the standard one for fermions on torus of size R with structure spin $\vec{\alpha}$

$$\begin{aligned} \langle \psi_L(\vec{x})\bar{\psi}_L(\vec{y}) \rangle \ &= \ \sum_{\vec{k}\in\mathbb{Z}^2} \frac{\exp\left(2\pi i(\vec{k}+\vec{\alpha})\cdot(\vec{x}-\vec{y})/R \right)}{i(k_z+\alpha_z)/R} \\ \langle \psi_R(\vec{x})\bar{\psi}_R(\vec{y}) \rangle \ &= \ -\sum_{\vec{k}\in\mathbb{Z}^2} \frac{\exp\left(2\pi i(\vec{k}+\vec{\alpha})\cdot(\vec{x}-\vec{y})/R \right)}{i(k_{\bar{z}}+\alpha_{\bar{z}})/R} \end{aligned} \tag{35}$$

where $k_z = k_1 - ik_2$ and $k_{\bar{z}} = k_1 + ik_2$. Finally, after a straightforward computation we get

$$\begin{aligned} &\langle \hat{S}_1(\vec{x})\ \hat{S}_2(\vec{y}) \rangle = \\ &-\sum_{\vec{k},\vec{k}'} \exp\left(\frac{4\pi^2\theta i}{R^2}\ (\vec{k}+\vec{\alpha}) \wedge (\vec{k}'+\vec{\alpha}) \right) \ \frac{\exp\left(2\pi i(\vec{k}-\vec{k}')\cdot(\vec{x}-\vec{y})/R \right)}{(k_z+\alpha_z)(k'_{\bar{z}}+\alpha_{\bar{z}})/R^2} \end{aligned} \tag{36}$$

Now let us compute the corresponding correlator in the dual torus $T'^2_{\theta=0}$. Using (18)-(20), the operators (32) are mapped to

$$\begin{aligned} S_1 \ &= \ \sum_{\vec{n},\vec{m}}^{N'-1} \bar{\chi}_R^{(\vec{n})}(x)\ \chi_L^{(\vec{m})}(x)\ J_{-2\vec{n}-2\vec{\alpha}} \cdot J_{2\vec{m}+2\vec{\alpha}} \\ S_2 \ &= \ \sum_{\vec{n},\vec{m}}^{N'-1} \bar{\chi}_L^{(\vec{n})}(x)\ \chi_R^{(\vec{m})}(x)\ J_{-2\vec{n}-2\vec{\alpha}} \cdot J_{2\vec{m}+2\vec{\alpha}} \end{aligned} \tag{37}$$

thus the v.e.v. is mapped to

$$\begin{aligned}
\langle S_1(x) \otimes S_2(y) \rangle &= \\
&= \sum_{\vec{n},\vec{m}}^{N'-1} \sum_{\vec{n}',\vec{m}'}^{N'-1} \langle \bar{\chi}_R^{(\vec{n})} \chi_L^{(\vec{m})} \bar{\chi}_L^{(\vec{n}')} \chi_R^{(\vec{m}')} \rangle \left(J_{-2\vec{n}-2\vec{\alpha}} \cdot J_{2\vec{m}+2\vec{\alpha}} \right) \otimes \left(J_{-2\vec{n}'-2\vec{\alpha}} \cdot J_{2\vec{m}'+2\vec{\alpha}} \right) \\
&= \sum_{\vec{n},\vec{n}'}^{N'-1} \exp\left(\frac{2\pi i M}{2N'} (2\vec{n}+2\vec{\alpha}) \wedge (2\vec{n}'+2\vec{\alpha}) \right) J_{2(\vec{n}-\vec{n}')} \otimes J_{2(\vec{n}'-\vec{n})} \times \\
&\qquad \langle \chi_L^{(\vec{n})}(\vec{x}) \bar{\chi}_L^{(\vec{n})}(\vec{y}) \rangle \, \langle \chi_R^{(\vec{n}')}(\vec{y}') \bar{\chi}_R^{(\vec{n}')}(\vec{x}) \rangle
\end{aligned} \tag{38}$$

The two point functions for the dual fermions are

$$\begin{aligned}
\langle \chi_L^{(\vec{n})}(\vec{x}) \bar{\chi}_L^{(\vec{n})}(\vec{y}) \rangle &= \sum_{\vec{k}\in\mathbb{Z}^2} \frac{\exp\left(2\pi N' i (\vec{k}+\vec{\alpha}) \cdot (\vec{x}-\vec{y})/R \right)}{i(k_z+\alpha_z)N'/R} \\
\langle \chi_R^{(\vec{n})}(\vec{x}) \bar{\chi}_R^{(\vec{n})}(\vec{y}) \rangle &= -\sum_{\vec{k}\in\mathbb{Z}^2} \frac{\exp\left(2\pi i N' (\vec{k}+\vec{\alpha}) \cdot (\vec{x}-\vec{y})/R \right)}{i(k_{\bar{z}}+\alpha_{\bar{z}})N'/R}
\end{aligned} \tag{39}$$

Finally, substituting the equations (39) into equation (38) and after a short computation we get

$$\begin{aligned}
\langle S_1(x) \otimes S_2(y) \rangle &= \\
&= -\sum_{\vec{k},\vec{k}'} \exp\left(\frac{4\pi^2 \theta i}{R^2} (\vec{k}+\vec{\alpha}) \wedge (\vec{k}'+\vec{\alpha}) \right) \frac{\exp\left(2\pi i (\vec{k}-\vec{k}') \cdot (\vec{x}-\vec{y})/R \right)}{(k_z+\alpha_z)(k'_{\bar{z}}+\alpha_{\bar{z}})/R^2} \times \\
&\qquad J_{2(\vec{k}-\vec{k}')} \otimes J_{2(\vec{k}'-\vec{k})}
\end{aligned} \tag{40}$$

This equation is identical to its noncommutative counterpart (36) after the substitution (15), showing the Morita equivalence of vacuum expectations values between dual theories. The generalization of this result to more complex correlation functions is straightforward. But once we have shown the equivalence of correlators in free theory, we can extend the results, perturbatively, to any "local" interaction of the fields. In fact, it is clear that the structure and singularities of dual correlators are identical, so the perturbative result to any order in dual theories will coincide.

4 Chiral anomaly and fermion determinant

So far we have studied the Morita equivalence between free fermion theories whose actions, being quadratic in the fields, are independent of θ. Even though this equivalence can be extended to arbitrary interactions in perturbation theory, one can still think that there is some degree of triviality in the examples arguing that we are just analyzing equivalences between theories defined around free actions. However the θ-independence manifests only at the level of the actions; the "local" operators of the theory, as $\hat{\hat{S}}_1$ and $\hat{\hat{S}}_2$ in (32), are of course θ dependent. In fact, for a fermion theory in noncommutative space, the "space of local interactions" in the sense of Wilson consists of all "local" star-product functionals of the fields.

Now let us study a less trivial example of the Morita equivalence, the chiral anomaly and its relation to bosonization. In particular, we are going to show the Morita equivalence between gauge effective actions of fermions coupled to a gauge field, which is clearly a non perturbative result.

Consider a theory of fermions in the noncommutative torus T^2_θ coupled minimally to a gauge field A_μ. In the noncommutative case there are three different $U(1)$ currents,

$$\begin{aligned} j^\mu_F &= \gamma^\mu_{ab}\ \psi_b * \bar{\psi}_a \\ j^\mu_A &= \gamma^\mu_{ab}\ \bar{\psi}_a * \psi_b \\ j^\mu_{Ad} &= \gamma^\mu_{ab}\ \left(\psi_b * \bar{\psi}_a + \bar{\psi}_a * \psi_b\right) \end{aligned} \tag{41}$$

and coupling the gauge field to each of them give rise to three different actions, usually called fundamental, antifundamental and adjoint respectively. We will work in the fundamental representation but the other cases are completely analogous. The action is

$$S = -\frac{1}{8\pi}\int_{T^2_\theta} d^2x\ \bar{\psi} * (\ \not\partial + \not A *)\ \psi + S[A_\mu] \tag{42}$$

where $S[A_\mu]$ is the gauge field action.

The gauge field satisfy periodic boundary conditions and can be expanded in Fourier modes as

$$A_\mu = \sum_{\vec{k}} A^{\vec{k}}_\mu U_{\vec{k}} \tag{43}$$

with $U_{\vec{k}}$ being Fourier modes defined in (5).

A chiral transformation takes the form [16], [17]

$$\psi'(x) = \mathcal{U}_5(x) * \psi \tag{44}$$

with

$$\mathcal{U}_5(x) = \exp_*(\gamma_5\alpha(x)) = 1 + \gamma_5\alpha + \frac{1}{2}\alpha(x) * \alpha(x) + \ldots \tag{45}$$

and leads to the anomalous conservation of the chiral current

$$\partial_\mu j^5_\mu = \mathcal{A}\ , \qquad j^5_\mu = \psi^T * (\gamma^5\gamma_\mu)^T\bar{\psi}^T \tag{46}$$

The chiral anomaly $\mathcal{A}$ can be calculated from the formula

$$\log J[\alpha] = -2\int_{T^2_\theta} d^2x\ \mathcal{A}\ * \delta\alpha, \tag{47}$$

$$\mathcal{A} = \mathrm{tr}\ \gamma_5\ \delta(x-x)|_{reg} \tag{48}$$

where $J[\alpha]$ is the Fujikawa Jacobian associated with an infinitesimal chiral transformation $\mathcal{U} = 1 + \gamma_5\delta\alpha$.

Before computing the anomaly, let us show its relation with the determinant of the Dirac operator. It is convenient to write the gauge field as

$$A_\mu = \frac{i}{e}\mathcal{U}[\phi,\eta] * \partial_\mu\mathcal{U}^{-1}[\phi,\eta] + A^0_\mu \tag{49}$$

with

$$\mathcal{U}[\phi,\eta] = \exp_*(\gamma_5\phi + i\eta) \tag{50}$$

and A^0_μ constants giving the Wilson phases around the cycles of the torus. Here we are assuming that, as it happens in ordinary space, a general gauge configuration can be parametrized as in (50).

We perform a change of the fermion variables

$$\begin{aligned} \psi &= \mathcal{U}_t * \chi_t\ , \\ \bar{\psi} &= \bar{\chi}_t * \mathcal{U}^\dagger_t \end{aligned} \tag{51}$$

where

$$\mathcal{U}_t = \exp_*\left(t\left(\gamma_5\phi + i\eta\right)\right)\ , \tag{52}$$

and t is a real parameter, $0 \leq t \leq 1$. In particular, if $t = 1$ the fermions decouple from the gauge fields up to the constant term A^0_μ. This change of variables has associated a Fujikawa jacobian $J[\phi,\eta;t]$ through the relation

$$\det{}_*\left(i\partial\!\!\!/ + eA\!\!\!/\right) = J[\phi,\eta;t]\,\det{}_*\left(i\ D\!\!\!\!/[t]\right) \tag{53}$$

where

$$D_\mu[t] = \partial_\mu + \mathcal{U}_{1-t} * \partial_\mu \mathcal{U}_{1-t}^{-1} - ieA_\mu^0 \tag{54}$$

Differentiating expression (53) with respect to t we have

$$\frac{d}{dt}\left(\log\ \det{}_*\left(i\, \not{D}[t]\right)\right) = -\frac{d}{dt}\left(\log J[\phi,\eta;t]\right) \tag{55}$$

and, after integrating in the range $0 \leq t \leq 1$, we obtain

$$\det{}_*\left(i\not{\partial} + e\not{A}\right) = \det{}_*\left(i\not{\partial} + e\not{A}^0\right)\exp\left(-2\int_0^1 dt \int_{T_\theta^2} d^2x\, \mathcal{A}(t) * \phi\right) \tag{56}$$

where we have identified

$$\int_{T_\theta^2} d^2x\, \mathcal{A}(t) * \phi = \frac{d}{dt}\left(\log_* J[\phi,\eta;t]\right) \tag{57}$$

Equations (55) and (56) relate the Fujikawa jacobian with the determinant of the Dirac operator and the chiral anomaly.

Now, expression (48) has to be regularized in a gauge-invariant way, thus as usual we write [16],[17]

$$\begin{aligned}
\mathcal{A}(t)_{reg} &= \lim_{M\to\infty} \mathrm{Tr}\left(\gamma_5 \exp_*\left(\frac{\not{D}[t]*\not{D}[t]}{M^2}\right)\right) \\
&= \lim_{M\to\infty} \mathrm{tr}\left(\gamma_5 \frac{1}{R^2}\sum_{\vec{k}} U_{\vec{k}+\vec{\alpha}}{}^\dagger \exp_*\left(\frac{\not{D}[t]*\not{D}[t]}{M^2}\right) U_{\vec{k}+\vec{\alpha}}\right)
\end{aligned} \tag{58}$$

and, after a straightforward calculation we obtain the usual result for the chiral anomaly

$$\mathcal{A}(t)_{reg} = \frac{e}{2\pi}\,\varepsilon^{\mu\nu} F_{\mu\nu}^t \tag{59}$$

where $F_{\mu\nu}^t$ is the electromagnetic field strength tensor

$$F_{\mu\nu}^t = \partial_\mu A_\nu^t - \partial_\nu A_\mu^t - ie(A_\mu^t * A_\nu^t - A_\mu^t * A_\nu^t) \tag{60}$$

and

$$A_\mu^t = \frac{i}{e}\mathcal{U}_{1-t}[\phi,\eta] * \partial_\mu \mathcal{U}_{1-t}^{-1}[\phi,\eta] \tag{61}$$

Finally, the fermion determinant takes the form

$$\det{}_*\,(i\partial\!\!\!/ + eA\!\!\!/) = \det{}_*\,\left(i\partial\!\!\!/ + eA\!\!\!/^0\right)\ \exp\left(-\frac{e}{2\pi}\int_{T^2_\theta} d^2x \int_0^1 dt\,\varepsilon^{\mu\nu}F^t_{\mu\nu} * \phi\right) \tag{62}$$

To compute the first factor we note the following. The constant field A^0_μ can be written as

$$A^0_\mu = \frac{i}{e}U_{\vec\beta} * \partial_\mu (U_{\vec\beta})^{-1} \tag{63}$$

where

$$U_{\vec\beta} \equiv \exp\left(2\pi i\vec\beta\cdot\vec x/R\right) \tag{64}$$

and

$$\vec\beta = \frac{R}{2\pi}\vec A_0 \tag{65}$$

Then, performing a gauge transformation to the fermions

$$\psi \to \psi' = U_{\vec\beta} * \psi \tag{66}$$

we can eliminate the constant gauge field from the determinant at the expense of changing the spin structures of the fermions

$$\vec\alpha \to \vec\alpha + \vec\beta. \tag{67}$$

Hence the determinant of the operator $i\partial\!\!\!/ + eA\!\!\!/^0$ is nothing but the partition function of a free fermion with spin structure $\vec\alpha + \vec\beta$

$$\det{}_*\left(i\partial\!\!\!/ + eA\!\!\!/^0\right) = \frac{\left|\vartheta\begin{bmatrix}\alpha_1+\beta_1-1/2\\ \alpha_2+\beta_2-1/2\end{bmatrix}(0,\tau)\right|^2}{|\eta(\tau)|^2} \tag{68}$$

At this point, we can compare the last result with the fermion determinant for the Morita-equivalent theory in the dual torus $T^{2'}$. The action (42) is mapped to the one

$$S = -\frac{N'}{16\pi}\mathrm{tr}_G\int_{T^{2'}} d^2x\ \bar\psi(\ \partial\!\!\!/ + A\!\!\!/)\ \psi + S[A_\mu] \tag{69}$$

where the fields ψ are $N\times N$ fermion-valued matrices and satisfy the boundary conditions (21). The $U(N)$ gauge field A_μ is defines on the dual torus $T_0^{\prime 2}$ and is given by [9]-[10].

$$A_\mu = \sum_{\vec{n}=0}^{N'-1} J_{\vec{n}} \sum_{\vec{q}\in\mathbb{Z}^2} \exp\left(2\pi i N'\vec{q}\cdot\vec{x}/R\right) A_\mu^{\vec{q},\vec{n}} U_{\vec{n}} \tag{70}$$

It satisfy the boundary conditions

$$\begin{aligned} A_\mu(x_1+R',x_2) &= \Omega_1^\dagger A_\mu(x_1,x_2)\Omega_1 \\ A_\mu(x_1,x_2+R') &= \Omega_2^\dagger A_\mu(x_1,x_2)\Omega_2 \end{aligned} \tag{71}$$

with Ω_1 and Ω_2 defined in equation (22). Notice that (71) are constant gauge transformations so the action is insensitive to them.

Using the mapped expressions for fermion fields, gauge fields and the t-dependent transformation, we can deal with this fermion determinant analogously. Indeed by a similar computation to the one used to obtain (56) we have

$$\det\left(i\partial\!\!\!/ + eA\!\!\!/\right) = \det\left(i\partial\!\!\!/ + eA\!\!\!/^0\right)\exp\left(-2\,\frac{1}{N}\mathrm{tr}_G\int_0^1 dt\int_{T^{2\prime}} d^2x\,\mathcal{A}(t)\phi\right) \tag{72}$$

where

$$\begin{aligned} \mathcal{A}(t)_{reg} &= \lim_{M\to\infty}\mathrm{Tr}\left(\gamma_5\exp\left(\frac{D\!\!\!\!/\,[t]^2}{M^2}\right)\right) \\ &= \lim_{M\to\infty}\mathrm{tr}\left(\gamma_5\frac{1}{R'^2}\sum_{\vec{q}}\sum_{\vec{n}=0}^{N'-1}\exp\left(-2\pi i(N'\vec{q}+\vec{n}+\vec{\alpha})\vec{x}/R\right)\ J^\dagger_{2\vec{n}+2\vec{\alpha}}\ \times\right. \\ &\qquad \left.\exp\left(\frac{D\!\!\!\!/\,[t]^2}{M^2}\right)\exp\left(2\pi i(N'\vec{q}+\vec{n}+\vec{\alpha})\vec{x}/R\right) J_{2\vec{n}+2\vec{\alpha}}\right) \end{aligned} \tag{73}$$

Finally, after a standard computation we get the well-known result for the nonabelian chiral anomaly in two dimensions, which inserted in equation (72) gives

$$\det\left(i\partial\!\!\!/ + eA\!\!\!/\right) = \det\left(i\partial\!\!\!/ + eA\!\!\!/^0\right)\exp\left(-\frac{eN'}{4\pi}\mathrm{tr}_G\int_{T^{2\prime}} d^2x\int_0^1 dt\,\varepsilon^{\mu\nu}F^t_{\mu\nu}\phi\right) \tag{74}$$

The first factor can be written analogously to the free fermion case as a product of N'^2 partition functions of free fermions with spin structures shifted by an amount $\vec{\beta}$, giving

$$\det\left(i\partial\!\!\!/ + eA\!\!\!/^0\right) = \prod_{n_1,n_2=0}^{N'-1}\left|\frac{\vartheta\begin{bmatrix}(\alpha_1+\beta_1+n_1)/N-1/2\\(\alpha_2+\beta_2+n_2)/N-1/2\end{bmatrix}(0,\tau)}{\eta(\tau)}\right|^2 \tag{75}$$

As it happens for the free fermion partition functions, it can be shown that (68) and (75) are identical.

Using relation (25) and the Morita mapping for the gauge fields it is immediate to show that the second factor in equation (74) is exactly the same as its noncommutative counterpart in equation (62). Thus, both determinants are equal.

We can go farther and perform the integration of the t-parameter in the anomaly equation (62). In fact, this integration can be done without difficult in the light-cone gauge using the parameterization

$$A^t_- = -\frac{i}{e} g * \partial_- g^{-1}\ ,\quad A^t_+ = 0 \tag{76}$$

$$g(\vec{x},t) = \exp_*(2\phi\, t) \tag{77}$$

After a trivial computation we get [16]-[17]

$$-\frac{e}{2\pi}\int_{T^2_\theta} d^2x \int_0^1 dt\, \varepsilon^{\mu\nu} F^t_{\mu\nu} * \phi = -\frac{1}{8\pi}\int_{T^2_\theta} d^2x \left(\partial_\mu g^{-1}\right) * (\partial_\mu g)$$
$$+\frac{i}{4\pi}\epsilon_{ij}\int_{T^2_\theta} d^2x \int_0^1 dt\; g^{-1} * (\partial_i g) * g^{-1} * (\partial_j g) * g^{-1} * (\partial_t g) \tag{78}$$

which is the Moyal deformation of the Wess-Zumino-Witten (WZW) action. This action is highly nonlinear and non-perturbative in nature. It is well known that, in ordinary space, the correlators of the WZW can be computed exactly, by invoking the infinite dimensional symmetries of the theory, namely the Virasoro algebra and the affine current algebra. However, it is not known how to solve this problem in noncommutative space as the star deformations of the Virasoro and affine algebras are not fully understood. Some progress in this direction was done in [17] where it was shown that, through a Seiberg-Witten mapping, the noncommutative WZW action is mapped to an ordinary space, $U(1)$ WZW model (a $U(1)$ WZW model in ordinary space is equivalent to a free masless boson theory). Notice nevertheless that the Seiberg-Witten mapping is not an isomorphism.

But in the noncommutative torus the situation is different; the Morita equivalence give us an isomorphism between noncommutative algebras and in the special case of a rational θ parameter, one of the isomorphic theories is a commutative one. Since we know that the gauge effective action of noncommutative theory of masless Dirac fermions coupled to a gauge field is a star deformed WZW theory and we have shown that, through the Morita equivalence, is mapped to an ordinary space WZW theory, we have an actual isomorphism between both theories. That is, we can give meaning to the concept of a "noncommutative conformal field theory", as the Morita equivalent version of an ordinary-space CFT.

Finally, it is not difficult to show [16], [18], [17] from eqs. (74),(78), that the bosonization of a free fermion theory is a star deformed WZW theory. It would be interesting to perform the Morita mapping to this theory and compare it with the bosonization of the ordinary space non-abelian fermion theory. We hope to report on this issue in a future work.

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